Fractals

Fractals are objects which:

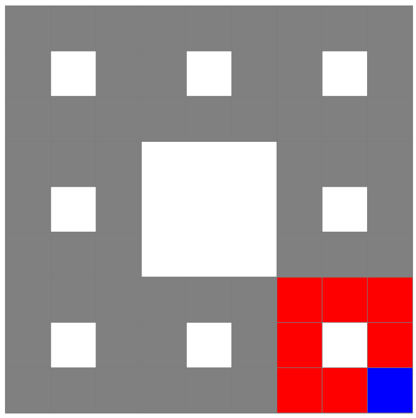
- if you zoom in and look at one part of it, it looks the same as the whole
- they have interesting structure, no matter how closely you look at them

Sierpinski Carpet

A nice example of a fractal is called the **Sierpinski Carpet**. It's made by taking a square, dividing it into nine smaller squares, and removing the central square.

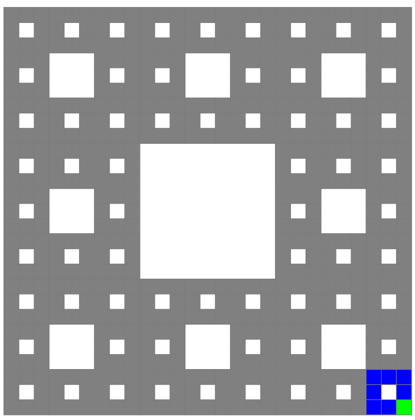
We can then repeat this process for the eight smaller squres left behind. We could then do this again to all of the even smaller remaining squares and so on. There will always be smaller squares left over.

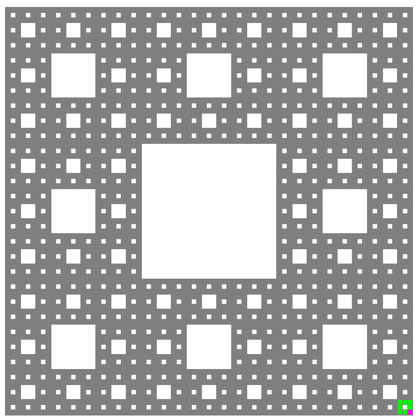
First stage



Second stage

Third stage





Fourth stage

If you repeat this process infinitely, the object you get is a **fractal**. If you look at any one part of it, the structure is the same as the whole set. Each of the smaller squares is a miniature version of the whole Sierpinski Carpet.

Because the process can go on for ever, no matter how much you zoom in on a Sierpinski Carpet, there will always be more pattern to see.

Cantor Set

We can make a one-dimensional version of the Sierpinski Carpet by taking a horizontal line, removing the middle third, and then repeating this for each of the two remaining segments.

If we continue to repeat the process forever, we get the **Cantor Set**. As before, to do this would take an infinitely long time, so we have to imagine the object we'd get.

Like before, once you have removed infinitely many stages, each smaller section of this line is the same as the whole line. The process breaks the original line up into infinitely many pieces.

Questions:

1. Starting with a complete square, and removing squares to make a Sierpinski carpet, after three steps what percentage of the area is left?

2. How could you calculate the total perimeter (inside and out) of the "fourth stage" Sierpinski Carpet?

   - Bonus: Calculate the total perimeter of the sixth stage Sierpinski Carpet.

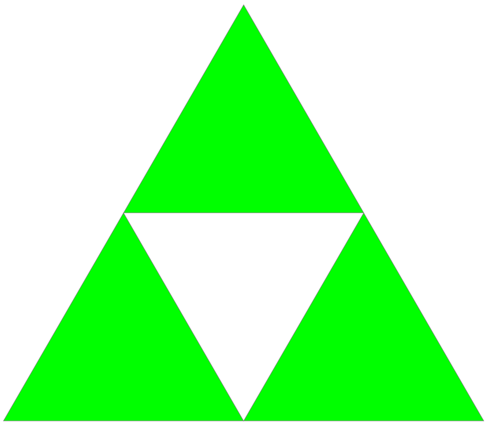
3. How many steps does it take until only one fifth of the Cantor Set is left?

4. In what way is this Romanesco broccoli (right) very like a fractal?

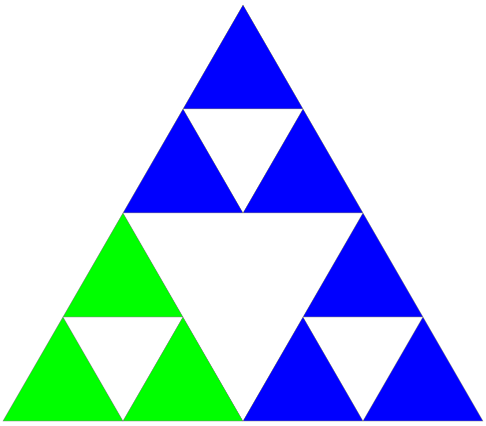
   Why is it not actually a true fractal?

Sierpinski Triangle

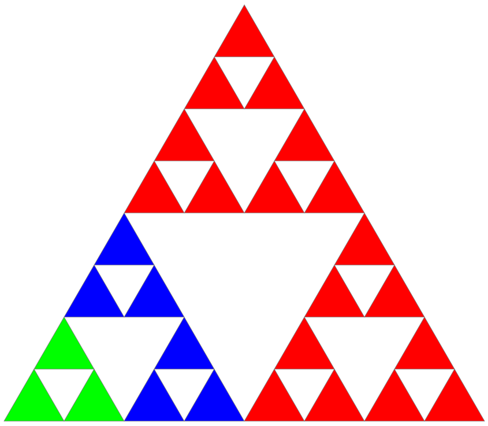
Another nice two-dimensional fractal can be made by splitting an equilateral triangle into four smaller ones, removing the middle one and repeating forever until you get a **Sierpinski Triangle**, sometimes also called a Sierpinski Sieve.



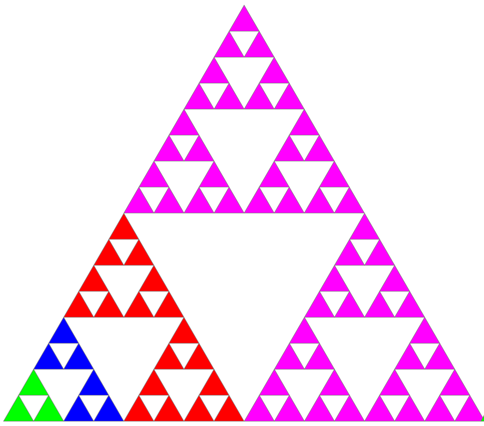
First stage



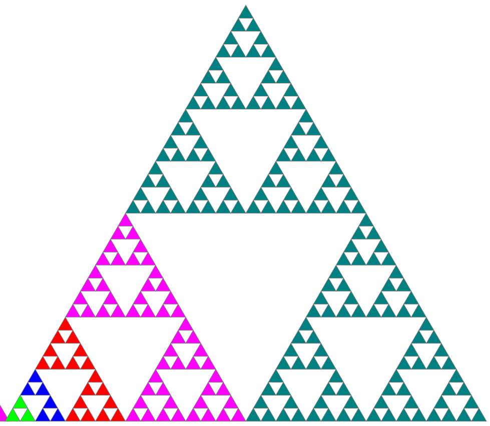
Second stage



Third stage



Fourth stage



Fifth stage...

Questions

1. How many triangles make up the second stage Sierpinski triangle?
2. How many more triangles are in the third stage than in the second stage?
3. Is there an easy way to calculate how many triangles there are in each stage?
4. What fraction of the area of the whole triangle are the smaller triangles which make up the first stage Sierpinski triangle? What about the second stage?

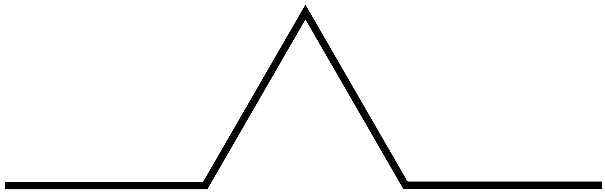
- Bonus: What fraction of the area of the triangle has been removed at Stage 3?

Koch Curve

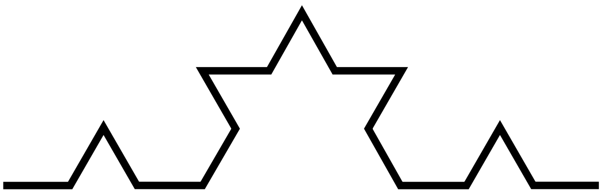
Instead of removing the middle third of a line, as for the Cantor Set, if you instead replace it with two sides of an equilateral triangle, and repeat for each section forever, you get the **Koch Curve**.

Each time you repeat, you increase the length of the line - so if you repeat forever, the line has infinite length!

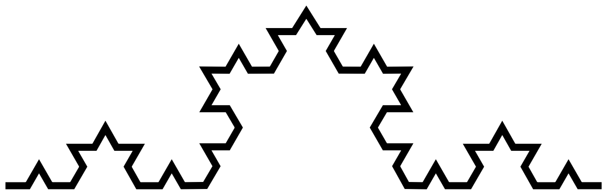
First stage



Second stage

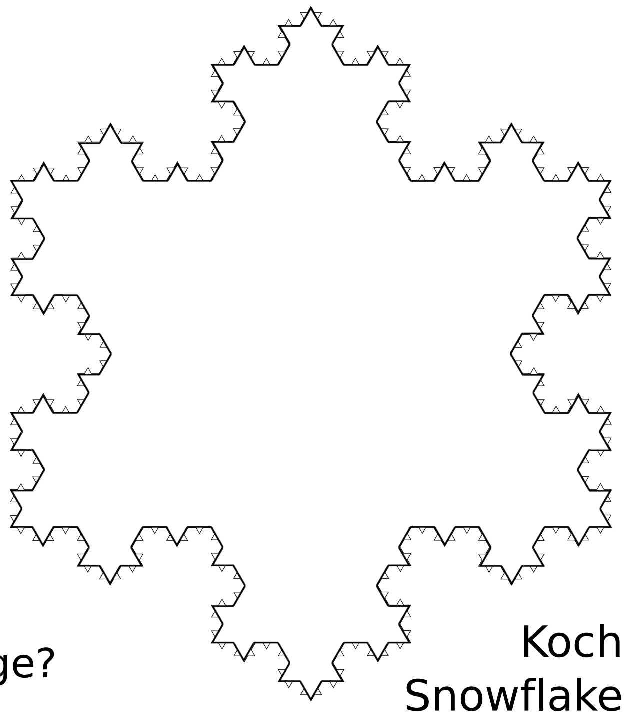


Third stage



Fourth stage

If instead of starting with a line, you started with an equilateral triangle, you get a **Koch Snowflake**.



Koch Snowflake

Questions

1. If the original first stage line was 12cm long, how long would the second stage line be? What about the third?

   Is there an easy way to calculate how the length increases at each stage?

2. How many triangles are added to the curve when you go from the 2nd stage to the 3rd stage? How many are added when you go from the 3rd to the 4th stage?

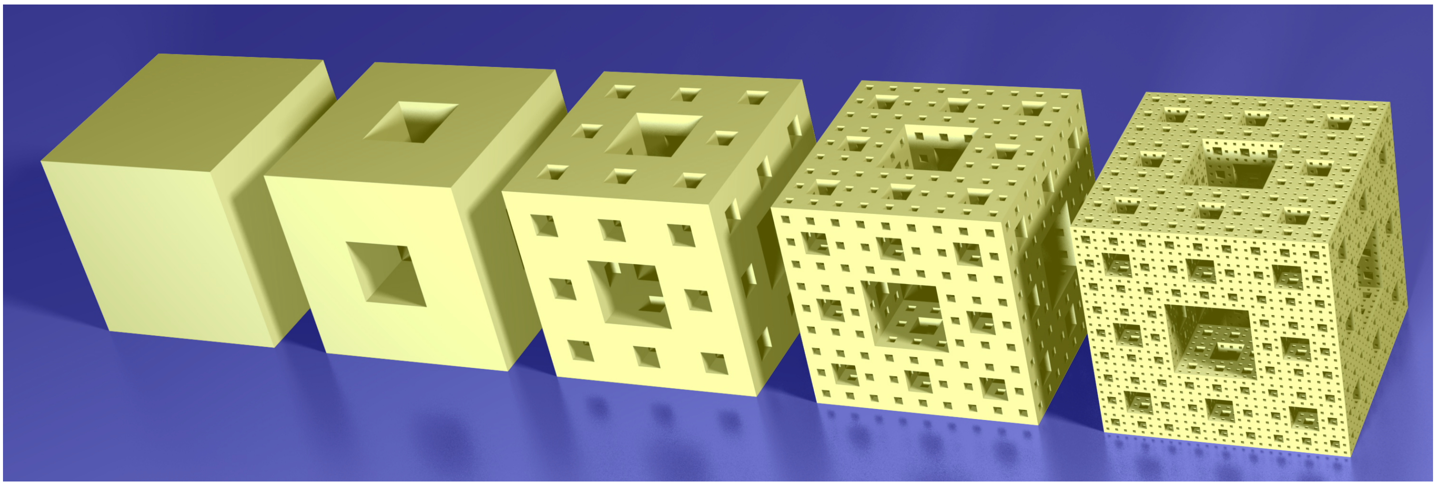
   Can you write down a way to calculate this number for any stage?

3D Fractals

There are lots of different three-dimensional fractals, many of which can be made using a repeated process.

Menger Sponge

A **Menger Sponge** is the three-dimensional version of the Sierpinski Carpet - it's made by splitting a cube into 27 smaller cubes, and removing 7 of them leaving a frame with a hole in each face. This is then repeated for each of the remaining smaller cubes, and so on.

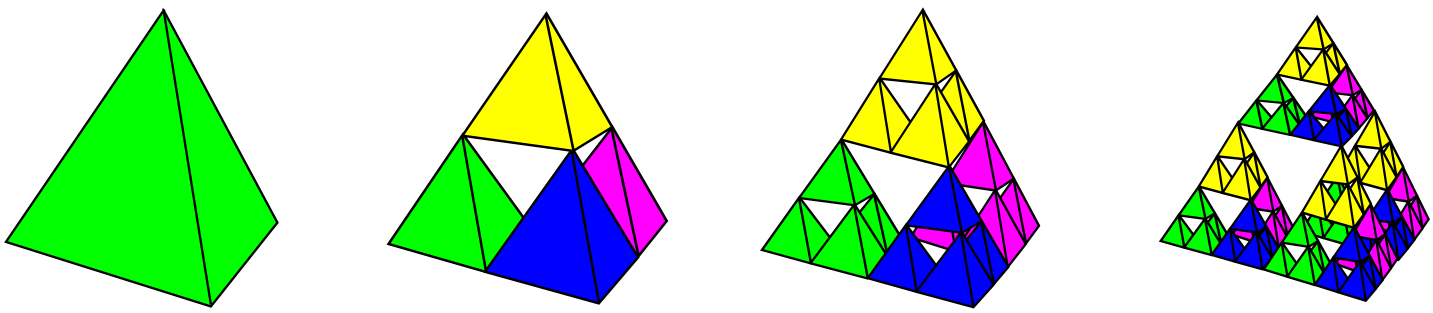


Construction of a Menger Sponge: based on an image by Niabot, CC-BY 3.0, via Wikimedia Commons

Sierpinski Tetrahedron

A **Sierpinski Tetrahedron**, sometimes also called a Tetrix, is a three- dimensional version of the Sierpinski triangle.

It's made by removing the middle section of a tetrahedron, leaving four smaller ones, and then repeating forever - or, alternatively, you can think of it as being made by joining together four smaller tetrahedra, then joining four of the larger ones, and so on.



Stage 1 (tetrahedron) Stage 2 Stage 3 Stage 4

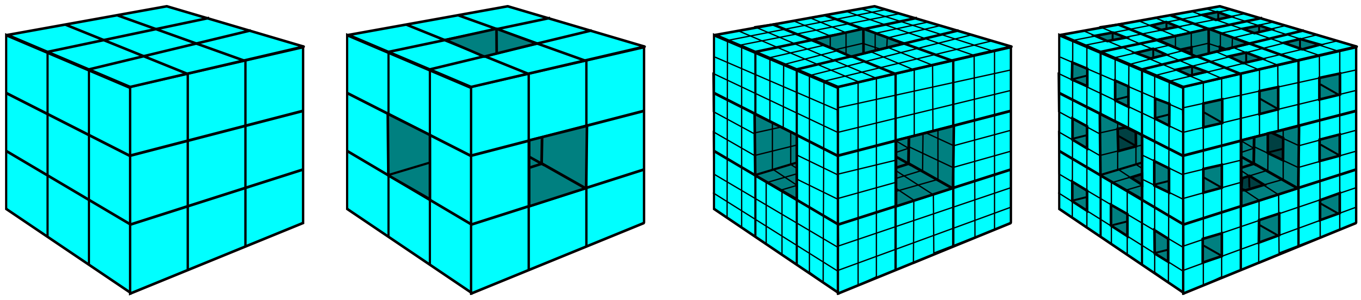
Questions

1. For the Menger Sponge:

   - How many small cubes are there in the first stage, after you remove one set of 7 cubes? If you split each of these cubes into smaller cubes, and remove 7 from each to make the stage 2 Menger Sponge, how many of these smaller cubes do you have?

   - If the whole cube measured 27cm along the edge, what is the length of the edge of one of the small cubes making up a stage 3 sponge?

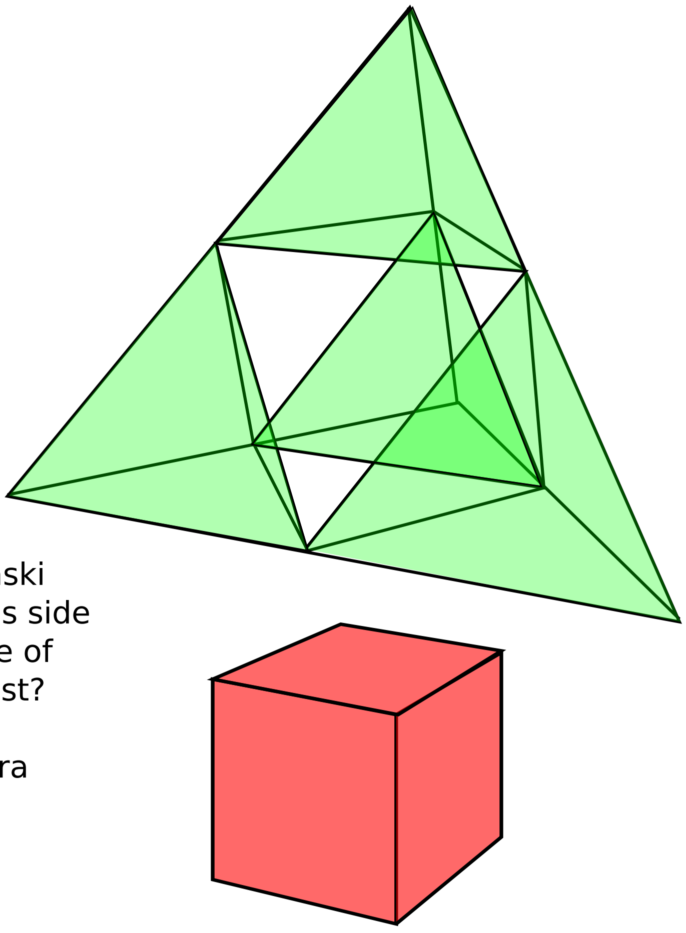
   - BONUS: What fraction of the volume of the whole cube is removed to go from Stage 1 to Stage 2? (Hint: It's not 7/27).



2. For the Sierpinski tetrahedron:

   - How many smaller tetrahedra are needed to make a stage 3 tetrix?
   - What about a stage 4 tetrix?

   - Is there an easy way to calculate how many smaller tetrahedra are needed to make any stage?



Bonus question:

If you wanted to balance a Sierpinski Tetrahedron on top of a Menger Sponge, which means the face of the Menger Sponge needs to be bigger than the hole in the bottom of the Sierpinski Tetrahedron, and your Menger Sponge has side length 21 centimetres, what does the side of your tetrahedron need to measure, at least?

If you're building it from smaller tetrahedra measuring 10cm along each edge, how many will you need?

Further Research:

- Who were Cantor, Sierpinski, Menger, and Koch? What can you find out about them?